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Relativistic quantum plasma dispersion functions

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Abstract

Relativistic quantum plasma dispersion functions (RQPDFs) are defined and the longitudinal and transverse response functions for an electron (plus positron) gas are written in terms of them. The dispersion is separated into Landau-damping, pair-creation and dissipationless regimes. Explicit forms are given for the RQPDFs in the cases of a completely degenerate distribution and a nondegenerate thermal (Jüttner) distribution. Particular emphasis is placed on the relation between dissipation and dispersion, with the dissipation treated in terms of the imaginary parts of RQPDFs. Comparing the dissipation calculated in this way with the existing treatments leads to the identification of errors in the literature, which we correct. We also comment on a controversy as to whether the dispersion curves in a superdense plasma pass through the region where pair creation is allowed.

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1. Introduction

There is a relatively sparse but diverse body of literature on relativistic quantum plasma dispersion functions (RQPDFs). Tsytovich (1961) calculated the response of a relativistic quantum (RQ) electron gas and derived expressions for the real and imaginary parts of the longitudinal and transverse response functions for arbitrary isotropic distributions and for a nondegenerate thermal (Jüttner) distribution. Jancovici (1962) derived expressions for the real and imaginary parts of the longitudinal and transverse response functions for a completely degenerate electron gas, providing a relativistic generalization of the well-known result of Lindhard (1954) for a nonrelativistic degenerate electron gas. These results were rederived and extended in various ways, by Hakim and Heyvaerts (1978, 1980) and Sivak (1985), using a Wigner function approach (Hakim 1978), by Delsante and Frankel (1980) and Kowalenko *et al* (1985), who concentrated on the longitudinal response, and by Hayes and Melrose (1984), who derived general results and Melrose and Hayes (1984), who extended Jancovici's (1962) results to the nearly degenerate limit and Tsytovich's (1961) results to a mildly degenerate plasma. The response functions were discussed further by Itoh *et al* (1992) and Braaten and Segel (1993) in connection with neutrino losses from stellar interiors. This area is of ongoing interest (Ratkoviċ *et al* 2003, Dutta *et al* 2004, Koers and Wijers 2005, Jaikumar *et al* 2005). One of the neutrino emission processes, the so-called plasma process, is dependent on dispersion and dissipation in the plasma, and the results of Braaten and Segel (1993) are used. Although these authors included RQ effects in their formal development, they approximated the general result, effectively by neglecting RQ effects in the resonant denominator, and it is their approximate form that is used. We argue that this leads to a misleading conclusion concerning one-photon pair creation (PC) in a superdense plasma, which we define here as plasmas with cutoff frequency (called the plasma frequency by some authors) exceeding the threshold, $2m$, for PC. (We use natural units with $\hbar = c = 1$.)

Our main purpose in this paper is to discuss the properties of RQPDFs in an isotropic, thermal RQ electron gas emphasizing the relation between dispersion and dissipation and the role of PC. One general feature of existing treatments is that the dispersion and dissipation are treated in different ways, with the dispersion described by appropriate RQPDFs, and with the dissipation calculated directly from the resonant part of the response function. In principle, the RQPDFs have imaginary parts that describe the dissipation, with the relation between the real and imaginary parts determined by the causal condition (e.g., the Landau criterion). Calculation of the dissipative part in this way provides a useful consistency check on the expression for the real part of the response function. Our consistency check fails in two published cases, and we identify and correct the relevant errors.

There are relatively few applications where the combination of intrinsically relativistic and quantum effects is important in a plasma. The extreme conditions required apply in, for example, the early universe, quark–gluon plasma and the interiors of compact stars. In section 6 we discuss possible applications, emphasizing a specific point relevant to the plasma process for neutrino emission: whether or not one-photon PC is possible in a superdense plasma. Earlier authors (Tsytovich 1961, Hayes and Melrose 1984, Kowalenko *et al* 1985) assumed that the dispersion curve does pass through the PC region, so that PC is allowed, and Itoh *et al* (1992) and Braaten and Segel (1993) gave arguments against this. Here we show that PC is possible in a superdense plasma and we determine the conditions under which it can occur.

In section 2 we present general formulae for the response functions and define general forms of RQPDFs for isotropic distributions. In section 3 we discuss dissipation, giving particular emphasis to the boundaries of the regions when Landau damping (LD) and PC are allowed for a given particle. In sections 4 and 5 we discuss the responses of a completely degenerate distribution and a nondegenerate thermal distribution, respectively. We discuss applications in section 6 and summarize our conclusions in section 7.

2. RQPDFs for isotropic distributions

An isotropic distribution is isotropic in one inertial frame, referred to as the rest frame of the medium. In a covariant formalism, which applies in an arbitrary frame, the motion of the rest frame relative to the origin in the arbitrary frame defines a 4-velocity, $\tilde{u}^\mu$. 'Choosing the rest frame' corresponds to setting $\tilde{u}^0 = 1$, $\tilde{\mathbf{u}} = 0$. The linear response tensor for an isotropic medium may be described in terms of the longitudinal and transverse response functions, which are written here in the arbitrary frame. In this section, we start with a general form for the response tensor, then write explicit forms for the longitudinal and transverse response functions and identify relevant RQPDFs.

2.1. General form for the response tensor

The general expression for the response tensor has been written in a variety of different forms. We start with a covariant form that is derived by analogy with the (unregularized) vacuum polarization tensor (e.g., Berestetskii *et al* (1971)), $\mathcal{P}^{\mu\nu}(k)$, which relates the Fourier transform in space and time of the linear induced 4-current, $J^{\mu}(k)$, to the 4-potential, $A^{\mu}(k)$, where k denotes the wave 4-vector, $[\omega, \mathbf{k}]$, constructed from the frequency, ω, and the wave 3-vector, $\mathbf{k}$. The response 4-tensor satisfies the charge-continuity and gauge-invariance relations, $k_{\mu}\mathcal{P}^{\mu\nu}(k) = 0$, $k_{\nu}\mathcal{P}^{\mu\nu}(k) = 0$. This form is

$$\mathcal{P}^{\mu\nu}(k) = -\mathrm{e}^2 \sum_{\zeta,\zeta'} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3} (2\pi)^3 \delta^3(\zeta'\mathbf{p}' - \zeta\mathbf{p} + \mathbf{k}) \times \frac{\frac{1}{2}(\zeta' - \zeta) + \zeta n^{\zeta}(\mathbf{p}) - \zeta' n^{\zeta'}(\mathbf{p}')}{\omega - \zeta\varepsilon + \zeta'\varepsilon'} \frac{F^{\mu\nu}(\zeta\tilde{p}, \zeta'\tilde{p}')}{\zeta\varepsilon\zeta'\varepsilon'}, \tag{1}$$

where $n^{\zeta}(\mathbf{p})$ is the occupation number, with $\zeta, \zeta' = \pm 1$ labelling electron and positron states, and with

$$F^{\mu\nu}(P, P') = P^{\mu}P'^{\nu} + P'^{\mu}P^{\nu} + g^{\mu\nu}(m^2 - PP'). \tag{2}$$

The energies are $\varepsilon = \varepsilon(\mathbf{p}) = (m^2 + |\mathbf{p}|^2)^{1/2}$, $\varepsilon' = \varepsilon(\mathbf{p}')$, and $\tilde{p}$ denotes the 4-momentum with components $[\varepsilon, \mathbf{p}]$. The (unregularized) vacuum polarization tensor itself follows from (1) by neglecting the contribution of the particles ($n^{\zeta}(\mathbf{p}) \to 0$).

The form (1) is general in the sense that it applies for an arbitrary distribution of unpolarized (or spin-averaged) electrons and positrons, subject only to the requirement that the distribution be stationary and uniform. No assumption is made about any background of ions. A distribution of ions would give an additional contribution to $\mathcal{P}^{\mu\nu}(k)$. It is well known that this contribution is important only at sufficiently low frequencies (below about the ion plasma frequency), and it is neglected here.

2.2. Longitudinal and transverse response functions

Isotropy implies that the response tensor is of the form

$$\mathcal{P}^{\mu\nu}(k) = \mathcal{P}^{L}(k)L^{\mu\nu}(k, \tilde{u}) + \mathcal{P}^{T}(k)T^{\mu\nu}(k, \tilde{u}), \tag{3}$$

where $L^{\mu\nu}(k, \tilde{u})$ and $T^{\mu\nu}(k, \tilde{u})$ are longitudinal and transverse projection operators in an arbitrary frame:

$$\begin{aligned} L^{\mu\nu}(k, u) &= \frac{k^2}{k^2 - (ku)^2}\left[a^{\mu\nu}(k, u) - \left(g^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2}\right)\right], \\ T^{\mu\nu}(k, u) &= \frac{1}{k^2 - (ku)^2}\left[-(ku)^2 a^{\mu\nu}(k, u) + k^2\left(g^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2}\right)\right], \\ a^{\mu\nu}(k, u) &= g^{\mu\nu} - \frac{k^{\mu}u^{\nu} + k^{\nu}u^{\mu}}{ku} + \frac{k^2 u^{\mu}u^{\nu}}{(ku)^2}. \end{aligned} \tag{4}$$

For a given $\mathcal{P}^{\mu\nu}(k)$ one may construct the longitudinal and transverse parts using

$$\mathcal{P}^{L}(k) = \frac{(k\tilde{u})^4}{k^4} L_{\mu\nu}(k, \tilde{u})\mathcal{P}^{\mu\nu}(k), \qquad \mathcal{P}^{T}(k) = \frac{1}{2} T_{\mu\nu}(k, \tilde{u})\mathcal{P}^{\mu\nu}(k). \tag{5}$$

Choosing the rest frame, this procedure reduces to the conventional 3-tensor procedure, which translates into

$$L^{ij}(k, \tilde{u}) = -\frac{k^i k^j}{|\mathbf{k}|^2}, \qquad T^{ij}(k, \tilde{u}) = g^{ij} + \frac{k^i k^j}{|\mathbf{k}|^2}, \tag{6}$$

$$\mathcal{P}^L(k) = \frac{k_i k_j}{|\mathbf{k}|^2}\mathcal{P}^{ij}(k), \qquad \mathcal{P}^T(k) = \frac{1}{2}\left(g_{ij} + \frac{k_i k_j}{|\mathbf{k}|^2}\right)\mathcal{P}^{ij}(k) \tag{7}$$

in terms of the space components of the 4-tensor notation for $\tilde{u}^\mu = [1, \mathbf{0}]$.

After projecting (1) to identify $\mathcal{P}^L(k)$ and $\mathcal{P}^T(k)$, one may perform the $\mathbf{p}'$ integral over the δ-function, choose the rest frame and rewrite the $\mathbf{p}$-integral in terms of integrals over ε, ε':

$$\int \mathrm{d}^3\mathbf{p} \to 2\pi \int_0^\infty \mathrm{d}|\mathbf{p}||\mathbf{p}|^2 \int_{-1}^1 \mathrm{d}\cos\theta = \frac{2\pi}{|\mathbf{k}|}\int_m^\infty \mathrm{d}\varepsilon\,\varepsilon \int_{\varepsilon'_{\min}}^{\varepsilon'_{\max}} \mathrm{d}\varepsilon'\varepsilon',$$

where the limiting values are

$$\varepsilon'_{\max,\min} = (\varepsilon^2 \pm 2|\mathbf{p}||\mathbf{k}| + |\mathbf{k}|^2)^{1/2}. \tag{8}$$

The response functions are unchanged by interchanging electrons and positrons. One may evaluate the response functions for electrons, and then replace the occupation number by

$$\tilde{n}(\varepsilon) = n^+(\varepsilon) + n^-(\varepsilon) \tag{9}$$

to include the contribution of the positrons.

Explicit forms for the response functions (neglecting the vacuum contribution) are (Hayes and Melrose 1984)

$$\mathcal{P}^L(k) = \frac{\mathrm{e}^2 n_{\mathrm{p}0}\omega^2}{m|\mathbf{k}|^2} + \frac{\mathrm{e}^2\omega^2 m}{8\pi^2|\mathbf{k}|^3}\left[(\omega^2 - |\mathbf{k}|^2)S^{(0)}(k) - 4m\omega S^{(1)}(k) + 4m^2 S^{(2)}(k)\right], \tag{10}$$

$$\mathcal{P}^T(k) = -\frac{\mathrm{e}^2 n_{\mathrm{p}0}(\omega^2 + |\mathbf{k}|^2)}{2m|\mathbf{k}|^2} - \frac{\mathrm{e}^2(\omega^2 - |\mathbf{k}|^2)m}{16\pi^2|\mathbf{k}|^3}\big[\big(-4\varepsilon_k^2 + \omega^2 + 2|\mathbf{k}|^2\big)S^{(0)}(k) - 4m\omega S^{(1)}(k) + 4m^2 S^{(2)}(k)\big], \tag{11}$$

with

$$\varepsilon_k = \frac{|\mathbf{k}|}{2}\left(\frac{\omega^2 - |\mathbf{k}|^2 - 4m^2}{\omega^2 - |\mathbf{k}|^2}\right)^{1/2}. \tag{12}$$

In (10), (11), the proper number density, $n_{\mathrm{p}0}$, appears: it is defined by

$$n_{\mathrm{p}0} = 2\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{m}{\varepsilon}\tilde{n}(\varepsilon), \qquad n = 2\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\tilde{n}(\varepsilon), \tag{13}$$

and it is an invariant, whereas the actual number density, n, depends on the choice of frame (and is the time component of a 4-vector). Also in (10), (11), three RQPDFs are introduced, $S^{(n)}(k)$, with $n = 0, 1, 2$. These RQPDFs involve integrals over the occupation number and two logarithmic functions:

$$S^{(n)}(k) = \int \frac{\mathrm{d}\varepsilon}{m}\left(\frac{\varepsilon}{m}\right)^n \tilde{n}(\varepsilon)\begin{cases}\ln\Lambda_1 & \text{for } n = 0, 2,\\ \ln\Lambda_2 & \text{for } n = 1,\end{cases} \tag{14}$$

$$\Lambda_1 = \frac{4\varepsilon^2\omega^2 - (\omega^2 - |\mathbf{k}|^2 - 2|\mathbf{p}||\mathbf{k}|)^2}{4\varepsilon^2\omega^2 - (\omega^2 - |\mathbf{k}|^2 + 2|\mathbf{p}||\mathbf{k}|)^2}, \qquad \Lambda_2 = \frac{4(\varepsilon\omega + |\mathbf{p}||\mathbf{k}|)^2 - (\omega^2 - |\mathbf{k}|^2)^2}{4(\varepsilon\omega - |\mathbf{p}||\mathbf{k}|)^2 - (\omega^2 - |\mathbf{k}|^2)^2}. \tag{15}$$

The response functions $\mathcal{P}^{L,T}$ are related to the dielectric response functions $\epsilon^{L,T}$ of Jancovici (1962) and Kowalenko *et al* (1985) as follows:

$$\mathrm{Im}\,\epsilon^{L,T} = \frac{\mu_0}{\omega^2}\,\mathrm{Im}\,\mathcal{P}^{L,T}, \qquad \mathrm{Re}\,\epsilon^{L,T} = 1 + \frac{\mu_0}{\omega^2}\,\mathrm{Re}\,\mathcal{P}^{L,T}, \tag{16}$$

resulting in dispersion relations in the rest frame of the plasma of the form

$$\omega^2 + \mu_0\,\mathrm{Re}\,\mathcal{P}^L(k) = 0, \qquad \omega^2 - |\mathbf{k}|^2 + \mu_0\,\mathrm{Re}\,\mathcal{P}^T(k) = 0. \tag{17}$$

(We use SI units, with μ_0 the permeability of free space: in Gaussian units one has $\mu_0 = 4\pi$.)

3. Dissipation

As already noted, in the existing literature it has been conventional to treat dissipation separately from dispersion, rather than treating it in terms of the imaginary parts of RQPDFs. Dissipation is possible only when ε_k, as given by (12), is real: LD applies for $\omega^2 - |\mathbf{k}|^2 < 0$ and PC for $\omega^2 - |\mathbf{k}|^2 > 4m^2$. Before discussing the conventional procedure, it is useful to identify the energy, momentum and speed of a resonant particle at the boundary of the allowed regions for LD or PC.

3.1. Limiting values of the resonance condition

In dispersion theory it is conventional to refer to a zero of the denominator (as a function of ω for fixed $\mathbf{k}$) as a resonance and to the algebraic condition for such a zero as a resonance condition. The Landau prescription specifies how one is to integrate around the associated pole in the integrand in accord with the causal condition, such that each pole contributes an imaginary part equal to its semi-residue. Dissipation is described by the contributions from these semi-residues, and different poles are interpreted in terms of different dissipation processes. There are two dissipation processes in a collisionless, unmagnetized plasma, LD and PC.

The resonance condition follows from conservation of 4-momentum on emission of a wave quantum, with p^μ and p'^μ the 4-momenta before and after the emission, respectively. Then one has $\zeta' p'^\mu = \zeta p^\mu - k^\mu = 0$, and squaring and requiring $p^2 = m^2$, $p'^2 = m^2$ gives the resonance condition $-2\zeta pk + k^2 = 0$, with $pk = \varepsilon\omega - \mathbf{p}\cdot\mathbf{k}$. For given ω, $|\mathbf{k}|$, the limiting values of the resonance condition correspond to $|\mathbf{p}\cdot\mathbf{k}| = |\mathbf{p}||\mathbf{k}|$, and these limiting values determine the boundaries of the regions where LD and PC are allowed. One may solve for the limiting values of ε, $|\mathbf{p}|$, $|\mathbf{v}| = |\mathbf{p}|/\varepsilon$, where $|\mathbf{v}|$ is the speed of the particle. Writing $|\mathbf{p}| = m\sinh\chi$, $\varepsilon = m\cosh\chi$, $|\mathbf{v}| = \tanh\chi$, $t = \tanh\left(\frac{1}{2}\chi\right)$, the limiting values must satisfy

$$(1+a)^2 t^4 + 2(1 - a^2 - 2b^2)t^2 + (1-a)^2 = 0, \qquad a = \frac{\omega^2 - |\mathbf{k}|^2}{2m\omega}, \quad b = \frac{|\mathbf{k}|}{\omega}. \tag{18}$$

The solutions for t^2 are

$$t^2 = t_\pm^2, \qquad t_\pm = \frac{b \pm (a^2 + b^2 - 1)^{1/2}}{1+a}. \tag{19}$$

There is considerable freedom in choosing the four solutions. An obvious choice is $t = \pm t_+, t = \pm t_-$. Noting that t and $-1/t$ correspond to the same values of ε, $|\mathbf{p}|$, $|\mathbf{v}|$, one may also choose from $t = \pm 1/t_+, t = \pm 1/t_-$. With $t = t_\pm$, the solutions for the energy, momentum and speed are

$$\frac{\varepsilon_\pm}{m} = \frac{a \pm b(a^2 + b^2 - 1)^{1/2}}{1 - b^2}, \qquad \frac{p_\pm}{m} = \frac{ab \pm (a^2 + b^2 - 1)^{1/2}}{1 - b^2},$$
$$v_\pm = \frac{p_\pm}{\varepsilon_\pm} = \frac{b \pm a(a^2 + b^2 - 1)^{1/2}}{a^2 + b^2}. \tag{20}$$

Another form of the boundary solutions was introduced by Tsytovich (1961); these are related to the $\pm$-solutions by

$$\varepsilon_\pm = \frac{1}{2}\omega \pm \varepsilon_k, \qquad p_\pm = \frac{1}{2}|\mathbf{k}| \pm \frac{\omega}{|\mathbf{k}|}\varepsilon_k, \tag{21}$$

with ε_k given by (12). The solutions (21) are the natural solutions for PC, and they correspond to the choice $t = t_\pm$. For LD, the natural solutions are $\varepsilon = \varepsilon_k \pm \frac{1}{2}\omega$, $|\mathbf{p}| = \omega\varepsilon_k/|\mathbf{k}| \pm \frac{1}{2}|\mathbf{k}|$, and these correspond to $t = t_-$ and $t = 1/t_+$, respectively. At a boundary of the LD or PC region, ε must correspond to either $\varepsilon_\pm$, with $\varepsilon'_{\rm max,min}$ then corresponding to $\varepsilon_\mp$, but some care is required in making the specific identifications.

3.2. Dissipation due to LD and PC

The resonant terms in (1) correspond to replacing the denominator by $-{\rm i}\pi\delta(\omega-\zeta\varepsilon+\zeta'\varepsilon')$, where the Landau prescription is used. We discuss LD and PC separately.

The terms with $\zeta = \zeta' = \pm 1$ describe LD, and these correspond to resonances at $\omega = \pm(\varepsilon-\varepsilon')$, respectively. On repeating the derivation of the longitudinal and transverse parts, the double integral over ε, ε' is reduced to a single integral by this δ-function, and the limit of integration can be expressed in terms of ε_k. In this way Tsytovich (1961) derived imaginary parts that correspond to

$$\mathrm{Im}\,\mathcal{P}^L_{\rm LD}(k) = \frac{{\rm e}^2\omega^2}{8\pi|\mathbf{k}|^3}\int_{\varepsilon_k}^{\infty}{\rm d}\varepsilon''(4\varepsilon''^2-|\mathbf{k}|^2)\left[\tilde n\left(\varepsilon''-\frac{1}{2}\omega\right)-\tilde n\left(\varepsilon''+\frac{1}{2}\omega\right)\right], \tag{22}$$

$$\mathrm{Im}\,\mathcal{P}^T_{\rm LD}(k) = \frac{{\rm e}^2(|\mathbf{k}|^2-\omega^2)}{8\pi|\mathbf{k}|^3}\int_{\varepsilon_k}^{\infty}{\rm d}\varepsilon''\big(2\varepsilon''^2-2\varepsilon_k^2+|\mathbf{k}|^2\big)\left[\tilde n\left(\varepsilon''-\frac{1}{2}\omega\right)-\tilde n\left(\varepsilon''+\frac{1}{2}\omega\right)\right], \tag{23}$$

in the notation used in this paper.

The resonant part of (1) describes PC for $\zeta\zeta' = -1$, but only $\zeta = 1 = -\zeta'$ contributes for $\omega > 0$. The vacuum contributes in this case, and Tsytovich (1961) retained both the contribution of the vacuum and of the electron gas, but his final expression for the vacuum contribution differs from the well-known result by a factor of 2. We note that the response is invariant under the interchange of positrons and electrons, which allows one to rewrite $1-n^+(\varepsilon)-n^-(\varepsilon')$ in (1) for $\zeta = 1 = -\zeta'$ as $1-\frac{1}{2}\tilde n(\varepsilon)-\frac{1}{2}\tilde n(\varepsilon')$. Then, repeating the derivation, we find

$$\mathrm{Im}\,\mathcal{P}^L_{\rm PC}(k) = -\frac{{\rm e}^2\omega^2}{8\pi|\mathbf{k}|^3}\int_{-\varepsilon_k}^{\varepsilon_k}{\rm d}\varepsilon''(4\varepsilon''^2-|\mathbf{k}|^2)\left[1-\frac{1}{2}\tilde n\left(\frac{1}{2}\omega+\varepsilon''\right)-\frac{1}{2}\tilde n\left(\frac{1}{2}\omega-\varepsilon''\right)\right], \tag{24}$$

$$\mathrm{Im}\,\mathcal{P}^T_{\rm PC}(k) = \frac{{\rm e}^2(\omega^2-|\mathbf{k}|^2)}{8\pi|\mathbf{k}|^3}\int_{-\varepsilon_k}^{\varepsilon_k}{\rm d}\varepsilon''\big(2\varepsilon''^2-2\varepsilon_k^2+|\mathbf{k}|^2\big)\times\left[1-\frac{1}{2}\tilde n\left(\frac{1}{2}\omega+\varepsilon''\right)-\frac{1}{2}\tilde n\left(\frac{1}{2}\omega-\varepsilon''\right)\right], \tag{25}$$

for the imaginary parts due to PC of the longitudinal and transverse responses, respectively. The unit term inside the square brackets differs by the relevant factor of 2 from Tsytovich's expression.

3.3. Dissipation due to the vacuum polarization tensor

The vacuum polarization tensor is of the form $\mathcal{P}_0^{\mu\nu}(k) = \mathcal{P}_0(k)(g^{\mu\nu}-k^\mu k^\nu/k^2)$, where $\mathcal{P}_0(k)$ is an invariant. The longitudinal and transverse parts are $\mathcal{P}_0^L(k) = \omega^2\mathcal{P}_0(k)/(\omega^2-|\mathbf{k}|^2)$ and $\mathcal{P}_0^T(k) = \mathcal{P}_0(k)$, respectively. The real part is negligible for most purposes involving wave dispersion, but the imaginary part cannot be neglected when considering dissipation due to PC. An important point is that dissipation due to PC occurs in the vacuum, and the presence of an electron gas tends to suppress it due to the Pauli exclusion principle. The imaginary part of the vacuum polarization tensor is well known, e.g., Berestetskii *et al* (1971), and corresponds to

$$\mathrm{Im}\,\mathcal{P}_0(k) = \frac{e^2}{3\pi}\left[m^2 + \frac{1}{2}(\omega^2 - |\mathbf{k}|^2)\right]\frac{\varepsilon_k}{|\mathbf{k}|}, \tag{26}$$

for $\omega^2 - |\mathbf{k}|^2 > 4m^2$, with $\mathrm{Im}\,\mathcal{P}_0(k) = 0$ for $\omega^2 - |\mathbf{k}|^2 < 4m^2$. A derivation of (26) using the approach adopted here leads to (24) and (25) with only the unit terms retained. The integrals are then elementary and (24) and (25) reproduce the longitudinal and transverse parts of (26), respectively.

3.4. Imaginary parts of logarithmic PDFs

Logarithmic functions appear naturally in (14), and more specifically as RQPDFs for a completely degenerate electron gas, as discussed below. It is desirable to have a prescription that allows one to write the imaginary part of a logarithmic PDF directly. Superficially, this seems trivial: when x passes from positive to negative, $\ln x$ may be replaced by $\ln|x| \pm \mathrm{i}\pi$. However, determining the relevant sign is not trivial.

The imaginary part of any PDF may be determined by imposing the Landau prescription. For a logarithmic PDF, this leads to the generic prescription

$$\mathrm{PDF}(\omega) = -\int_{\omega_{\min}}^{\omega_{\max}} \frac{\mathrm{d}x}{\omega - x + \mathrm{i}0} = \begin{cases} \ln\dfrac{\omega-\omega_{\max}}{\omega-\omega_{\min}} & \text{for } \omega < \omega_{\min}, \quad \omega > \omega_{\max}, \\ \ln\left|\dfrac{\omega-\omega_{\max}}{\omega-\omega_{\min}}\right| + \mathrm{i}\pi & \text{for } \omega_{\min} < \omega < \omega_{\max}. \end{cases} \tag{27}$$

Hence, to impose the causal condition, one writes a logarithmic function as a sum (or difference) of terms of the form (27) and gives each term an imaginary part, $\mathrm{i}\pi$, when the frequency is in the range $\omega_{\min} < \omega < \omega_{\max}$.

4. Completely degenerate Fermi gas

The response of a completely degenerate Fermi gas was calculated by Jancovici (1962), cf also Hayes and Melrose (1984), Sivak (1985), Kowalenko *et al* (1985). Jancovici's expression for the transverse part of the response tensor contains a spurious factor $\omega^2/(\omega^2 - |\mathbf{k}|^2)$, which leads to a nonphysical resonance in the dispersion relation for transverse waves. It also implies incorrectly that the contribution of the electron gas to dissipation due to LD and PC has the same sign for the transverse response. Here we start with forms that are valid in the DL region, where the response functions are necessarily real, and then discuss the extensions into the LD and PC regions.

4.1. Thermal distributions

Before considering the completely degenerate limit, it is appropriate to comment on the general case of a thermal distribution of electrons, which is the Fermi–Dirac distribution,

$$\tilde{n}(\varepsilon) = \frac{1}{\exp[(\varepsilon-\mu)/T]+1} + \frac{1}{\exp[(\varepsilon+\mu)/T]+1}, \tag{28}$$

where the chemical potential, μ, includes the rest energy, m, and where the temperature is T, with Boltzmann's constant set to unity. (The form (28) applies for each spin state, and a factor of 2 arises when one sums over the two spin states for unpolarized electrons and positrons.)

No general results are known for the RQPDFs for this case. The completely degenerate limit corresponds to $T \to 0$, $\mu \to \varepsilon_F$, when one has

$$\tilde{n}(\varepsilon) = \begin{cases} 1 & \text{for } \varepsilon < \varepsilon_F, \\ 0 & \text{for } \varepsilon > \varepsilon_F, \end{cases} \tag{29}$$

where $\varepsilon_F = \left(m^2 + p_F^2\right)^{1/2}$ is the Fermi energy, with the Fermi momentum determined by the electron number density, $n_e = p_F^3/3\pi^2$.

4.2. DL region

In the DL region Jancovici's response functions, in the present notation (with the spurious factor omitted), are

$$\begin{aligned}\mathcal{P}^L(k) = \frac{e^2\omega^2}{4\pi^2|\mathbf{k}|^2}\Bigg\{ &\frac{8\varepsilon_F p_F}{3} - \frac{2|\mathbf{k}|^2}{3}\ln\left(\frac{\varepsilon_F + p_F}{m}\right) + \frac{\varepsilon_F\left[4\varepsilon_F^2 + 3(\omega^2 - |\mathbf{k}|^2)\right]}{6|\mathbf{k}|}\ln\Lambda_{1F} \\ &+ \frac{\omega\left[3|\mathbf{k}|^2 - \omega^2 - 12\varepsilon_F^2\right]}{12|\mathbf{k}|}\ln\Lambda_{2F} + \frac{2m^2 + \omega^2 - |\mathbf{k}|^2}{3(\omega^2 - |\mathbf{k}|^2)}|\mathbf{k}|\varepsilon_k\frac{\omega}{|\omega|}\ln\Lambda_{3F}\Bigg\},\end{aligned} \tag{30}$$

$$\begin{aligned}\mathcal{P}^T(k) = -\frac{e^2}{4\pi^2}\Bigg\{ &\frac{4\omega^2 + 2|\mathbf{k}|^2}{3|\mathbf{k}|^2}\varepsilon_F p_F + \frac{2(\omega^2 - |\mathbf{k}|^2)}{3}\ln\left(\frac{\varepsilon_F + p_F}{m}\right) \\ &+ \varepsilon_F\left[\frac{\varepsilon_F^2(\omega^2 - |\mathbf{k}|^2)}{3|\mathbf{k}|^3} + \frac{4m^2|\mathbf{k}|^2 + \omega^4 - |\mathbf{k}|^4}{4|\mathbf{k}|^3}\right]\ln\Lambda_{1F} \\ &- \frac{\omega\left[12m^2|\mathbf{k}|^2 + (\omega^2 - |\mathbf{k}|^2)\left(12\varepsilon_F^2 + \omega^2 + 3|\mathbf{k}|^2\right)\right]}{24|\mathbf{k}|^3}\ln\Lambda_{2F} \\ &- \frac{2m^2 + \omega^2 - |\mathbf{k}|^2}{3|\mathbf{k}|}\varepsilon_k\frac{\omega}{|\omega|}\ln\Lambda_{3F}\Bigg\},\end{aligned} \tag{31}$$

with Λ_{iF} given by setting $|\mathbf{p}| = p_F$, $\varepsilon = \varepsilon_F$ in expressions (15) for Λ_1, Λ_2 and

$$\Lambda_3 = \frac{(\omega^2 - |\mathbf{k}|^2)^2(\varepsilon|\mathbf{k}| + 2|\mathbf{p}|\varepsilon_k)^2 - 4m^4\omega^2|\mathbf{k}|^2}{(\omega^2 - |\mathbf{k}|^2)^2(\varepsilon|\mathbf{k}| - 2|\mathbf{p}|\varepsilon_k)^2 - 4m^4\omega^2|\mathbf{k}|^2}. \tag{32}$$

Note that the sign $\omega/|\omega|$ is needed in the terms involving Λ_{3F} to ensure that the real parts of the response functions are even functions of ω.

The DL regime corresponds to $|\mathbf{k}|^2 < \omega^2 < 4m^2 + |\mathbf{k}|^2$. In this regime, Λ_{1F} and Λ_{2F} are positive (they are always real) so their logarithms are real. Due to ε_k being imaginary, Λ_{3F} is the ratio of a complex number and its complex conjugate, and hence its logarithm is imaginary, so that $\varepsilon_k \ln\Lambda_{3F}$ is real. One may write

$$\varepsilon_k \ln\Lambda_{3F} = -2|\varepsilon_k| \begin{cases} \arctan\chi_+ - \arctan\chi_- & \text{for } \chi_- > 0, \\ \arctan\chi_+ - \arctan\chi_- - \pi & \text{for } \chi_- < 0, \end{cases} \tag{33}$$

$$\chi_\pm = \frac{2\varepsilon_F(\omega^2 - |\mathbf{k}|^2)|\varepsilon_k|}{|\mathbf{k}|[2m^2|\mathbf{k}| \pm p_F(\omega^2 - |\mathbf{k}|^2)]}. \tag{34}$$

Although earlier authors, e.g., Kowalenko *et al* (1985), Sivak (1985), noted that $\ln\Lambda_{3F}$ is replaced by an arctangent, there are many choices as to how this arctangent is written. With the choice made in (33) the arctangents $\chi_\pm$ remain between 0 and $\pi/2$ throughout the DL range. The choice (34) avoids complications with other choices in numerical calculations.

Table 1. Imaginary parts of RQPDFs for a completely degenerate electron gas.

		Upper boundaries	$\ln\Lambda_1$	$\ln\Lambda_2$	$\ln\Lambda_3$
(c)	LD	$\omega < \varepsilon_F - \varepsilon'_{Fmin}$, $\lvert\mathbf{k}\rvert < 2p_F$	0	$-i2\pi$	0
(b)	LD	$\lvert\varepsilon_F - \varepsilon'_{Fmin}\rvert < \omega < \varepsilon'_{Fmax} - \varepsilon_F$	$i\pi$	$-i\pi$	$i\pi$
(e)	PC	$(4m^2 + \lvert\mathbf{k}\rvert^2)^{1/2} < \omega < \varepsilon_F + \varepsilon'_{Fmin}$, $\lvert\mathbf{k}\rvert < 2p_F$	0	0	$-2i\pi$
(f)	PC	$\varepsilon_F + \varepsilon'_{Fmin} < \omega < \varepsilon_F + \varepsilon'_{Fmax}$	$i\pi$	$i\pi$	$-i\pi$

4.3. *Imaginary of* $\ln\Lambda_{iF}$

In order to use the prescription (27), the logarithmic functions must be written in an appropriate form. Relevant forms for Λ_1, Λ_2 follow by writing (15) in terms of the limiting values (8). This gives

$$\Lambda_1 = \frac{(\omega - \varepsilon + \varepsilon'_{max})(\omega + \varepsilon - \varepsilon'_{max})(\omega - \varepsilon - \varepsilon'_{max})(\omega + \varepsilon + \varepsilon'_{max})}{(\omega - \varepsilon + \varepsilon'_{min})(\omega + \varepsilon - \varepsilon'_{min})(\omega - \varepsilon - \varepsilon'_{min})(\omega + \varepsilon + \varepsilon'_{min})}, \tag{35}$$

$$\Lambda_2 = \frac{(\omega - \varepsilon + \varepsilon'_{max})(\omega - \varepsilon - \varepsilon'_{max})(\omega + \varepsilon - \varepsilon'_{min})(\omega + \varepsilon + \varepsilon'_{min})}{(\omega - \varepsilon + \varepsilon'_{min})(\omega - \varepsilon - \varepsilon'_{min})(\omega + \varepsilon - \varepsilon'_{max})(\omega + \varepsilon + \varepsilon'_{max})}. \tag{36}$$

Although Λ_3 cannot be rewritten in terms of the factors that appear in (35) and (36), it can be written in a form similar to (36), with ω replaced by $2\varepsilon_k$:

$$\Lambda_3 = \frac{(2\varepsilon_k - \varepsilon + \varepsilon'_{max})(2\varepsilon_k - \varepsilon - \varepsilon'_{max})(2\varepsilon_k + \varepsilon - \varepsilon'_{min})(2\varepsilon_k + \varepsilon + \varepsilon'_{min})}{(2\varepsilon_k - \varepsilon + \varepsilon'_{min})(2\varepsilon_k - \varepsilon - \varepsilon'_{min})(2\varepsilon_k + \varepsilon - \varepsilon'_{max})(2\varepsilon_k + \varepsilon + \varepsilon'_{max})}. \tag{37}$$

The logarithms of Λ_1, Λ_2 may be written as a sum of terms of the form (27) by identifying $(\omega - \omega_{max})/(\omega - \omega_{min})$ with $(\omega \pm \varepsilon - \varepsilon'_{max})/(\omega \pm \varepsilon - \varepsilon'_{min})$ or with $(\omega \pm \varepsilon + \varepsilon'_{min})/(\omega \pm \varepsilon + \varepsilon'_{max})$. Although $\ln\Lambda_3$ cannot be rewritten in terms of these factors, in the neighbourhood of the zeros of any of the factors in (37), the vanishing factor does become of this form. As only the sign of the imaginary part on crossing the zero is required, this suffices to determine the sign. For example, consider the factor $(2\varepsilon_k - \varepsilon - \varepsilon'_{max})$: this factor is zero at the boundary of the LD region with $\varepsilon = \varepsilon_k - \frac{1}{2}\omega$, $\varepsilon'_{max} = \varepsilon_k + \frac{1}{2}\omega$, and in the neighbourhood of this boundary the factor may be approximated by $(\omega - \varepsilon - \varepsilon'_{max})$, and treated in the same manner as the corresponding factor in Λ_1 or Λ_2.

The boundaries of the allowed regions for LD and PC are illustrated in figure 1. For $|\mathbf{k}| > 2p_F$, one has $\varepsilon_F < \varepsilon'_{Fmin}$; then the upper and lower frequency boundaries are $\omega = \varepsilon'_{Fmax,min} - \varepsilon_F$ for LD, and $\omega = \varepsilon'_{Fmax,min} + \varepsilon_F$ for PC. In this case only $\ln[(\omega \pm \varepsilon_F - \varepsilon'_{Fmax})/(\omega \pm \varepsilon_F - \varepsilon'_{Fmin})]$ contribute to LD and PC, respectively, with these factors giving an imaginary part of $i\pi$ in the region where the argument of the logarithm is negative, and zero otherwise. For $|\mathbf{k}| < 2p_F$ one has $\varepsilon_F > \varepsilon'_{Fmin}$. In this case, the zero of $\omega - \varepsilon_F + \varepsilon'_{Fmin}$ occurs within the LD region, separating regions (b) and (c) in figure 1, and the zero of $\omega - \varepsilon_F - \varepsilon'_{Fmin}$ occurs within the PC region, separating regions (e) and (f) in figure 1. It is then straightforward to determine the signs of the imaginary parts in the various regions, and these are listed in table 1.

The imaginary parts of $\mathcal{P}^L(k)$ and $\mathcal{P}^T(k)$ may be written by inspection using (30) and (31), respectively, and noting the imaginary parts in table 1. These imaginary parts may also be derived from (22) and (23) by setting the occupation number equal to unity for $\varepsilon < \varepsilon_F$ and zero for $\varepsilon > \varepsilon_F$, and performing the integrals, which are then elementary. The imaginary part

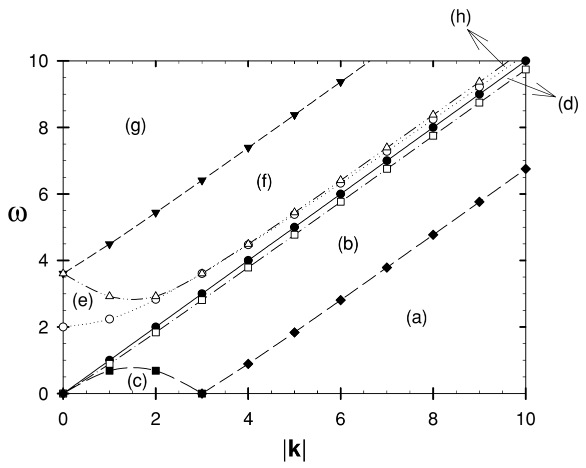

Figure 1. Regions of ω–$|\mathbf{k}|$ space ($\omega > 0$) are separated by curves corresponding to the boundaries of the regions where LD is allowed, $\omega < |\mathbf{k}|$, and PC is allowed, $\omega > (4m^2 + |\mathbf{k}|^2)^{1/2}$. These are further separated into regions (a)–(h) defined for a completely degenerate electron gas with $p_F/m = 1.5$. For the completely degenerate gas, LD is allowed only in regions (b) and (c), and there is no dissipation in (a) and (d), the electron gas completely suppresses PC in (e) and partly suppresses PC in (f); PC has its vacuum value in (g) and (h). From lower right to upper left the curves are: $\omega = (\varepsilon_F^2 - 2p_F|\mathbf{k}| + |\mathbf{k}|^2)^{1/2} - \varepsilon_F$ (dashed, solid diamonds), $\omega = \varepsilon_F - (\varepsilon_F^2 - 2p_F|\mathbf{k}| + |\mathbf{k}|^2)^{1/2}$ (dashed, solid squares), $\omega = (\varepsilon_F^2 + 2p_F|\mathbf{k}| + |\mathbf{k}|^2)^{1/2} - \varepsilon_F$ (dot-dashed, open squares), $\omega = |\mathbf{k}|$ (solid, solid circles), $\omega = (4m^2 + |\mathbf{k}|^2)^{1/2}$ (dotted, open circles), $\omega = (\varepsilon_F^2 - 2p_F|\mathbf{k}| + |\mathbf{k}|^2)^{1/2} + \varepsilon_F$ (double-dot-dashed, open triangles), $\omega = (\varepsilon_F^2 + 2p_F|\mathbf{k}| + |\mathbf{k}|^2)^{1/2} + \varepsilon_F$ (dashed, solid triangles).

was written for the longitudinal response by Jancovici (1962) and Kowalenko *et al* (1985) and these results are reproduced using (30) and table 1.

4.4. Alternative forms for Λ_i

For completeness we note that the logarithmic functions may be written in terms of the $\pm$ solutions (19) and (20):

$$\begin{aligned}
\Lambda_1 &= \frac{(t+t_+)(t+t_-)(t-1/t_+)(t-1/t_-)}{(t-t_+)(t-t_-)(t+1/t_+)(t+1/t_-)} = \frac{(|\mathbf{p}|+p_+)(|\mathbf{p}|+p_-)}{(|\mathbf{p}|-p_+)(|\mathbf{p}|-p_-)}, \\
\Lambda_2 &= \frac{(t+t_+)(t+t_-)(t+1/t_+)(t+1/t_-)}{(t-t_+)(t-t_-)(t-1/t_+)(t-1/t_-)} = \frac{(|\mathbf{v}|+v_+)(|\mathbf{v}|+v_-)}{(|\mathbf{v}|-v_+)(|\mathbf{v}|-v_-)}, \qquad (38)\\
\Lambda_3 &= \frac{(t+t_+)(t-t_-)(t+1/t_+)(t-1/t_-)}{(t-t_+)(t+t_-)(t-1/t_+)(t+1/t_-)} = \frac{(|\mathbf{v}|+v_+)(|\mathbf{v}|-v_-)}{(|\mathbf{v}|-v_+)(|\mathbf{v}|+v_-)}.
\end{aligned}$$

The boundaries in figure 1 are identified as follows: the upper boundary to the PC region corresponds to $t_F = -t_-$ ($p_F = -p_-, v_F = -v_-$), the lower boundary of region (f) corresponds to $t_F = t_-$ ($p_F = p_-, v_F = v_-$) for $|\mathbf{k}| > 2p_F$ and to $t_F = t_+$ ($p_F = p_+, v_F = v_+$) for $|\mathbf{k}| < 2p_F$; the upper boundary of region (b) corresponds to $t_F = 1/t_+$ ($p_F = -p_+$, $v_F = v_+$), the lower boundary corresponds to $t_F = -1/t_+$ ($p_F = p_+, v_F = -v_+$) for $|\mathbf{k}| > 2p_F$ and to $t_F = t_-$ ($p_F = p_-, v_F = v_-$) for $|\mathbf{k}| < 2p_F$. Although useful for some other purposes, these forms are not convenient for determining the imaginary parts because the frequency dependence is implicit rather than explicit, and the prescription (27) cannot be used directly.

5. Nondegenerate thermal distribution

The nondegenerate limit of the Fermi–Dirac distribution (28) applies when μ/T is large and negative, and then it becomes the Jüttner distribution,

$$\tilde{n}(\varepsilon) = A\,\mathrm{e}^{-\varepsilon/T}, \qquad A = 2\cosh(\mu/T) = \frac{\pi^2\tilde{n}_e}{2m^2 T K_2(m/T)}, \tag{39}$$

where K_2 is a modified Bessel function, and $\tilde{n}_e$ is the number density of electrons plus positrons. (The normalization coefficient, A, is evaluated by setting the integral of $\tilde{n}(\varepsilon)$ over $2\mathrm{d}^3\mathbf{p}/(2\pi)^3$ equal to $\tilde{n}_e$, where the factor 2 arises from the sum over the two spin states.)

According to Melrose and Hayes (1984), in this case the three plasma dispersion functions $S^{(n)}(k)$ can be evaluated in terms of the RPDF introduced by Godfrey *et al* (1975), which they wrote in the form

$$T(z,\rho) = \int_{-1}^{1} \frac{\mathrm{d}v}{v-z}\exp(-\rho\gamma), \tag{40}$$

where $z = \omega/|\mathbf{k}|$ is the phase speed, $\gamma = (1-v^2)^{-1/2}$ and $\rho = m/T$ is an inverse temperature in units of the rest energy of the electron ($\rho = 1$ corresponds to $T = 0.5\times10^{10}$K). Dissipation is described by the imaginary part of this RPDF:

$$\mathrm{Im}\,T(z,\rho) = \begin{cases} \pi\,\mathrm{e}^{-\rho\gamma_0} & \text{for } |z|<1, \\ 0 & \text{for } |z|>1, \end{cases} \tag{41}$$

with $\gamma_0 = (1-z^2)^{-1/2}$. The RQPDFs become

$$\begin{aligned}
S^{(0)}(k) &= \frac{A}{\rho}\sum_{\pm}\frac{\sigma_\pm}{\gamma_\pm v_\pm}\left(\frac{1-v_\pm^2}{\rho}T'(v_\pm,\rho) + 2K_1(\rho)\right), \\
S^{(1)}(k) &= \frac{A}{\rho}\sum_{\pm}\left[-\frac{T(v_\pm,\rho)}{\rho} + \frac{1}{v_\pm}\left(\frac{1-v_\pm^2}{\rho}T'(v_\pm,\rho) + 2K_1(\rho)\right)\right], \\
S^{(2)}(k) &= \frac{A}{\rho}\sum_{\pm}\frac{\sigma_\pm}{\gamma_\pm v_\pm}\left\{\left(\frac{2}{\rho^2}+\gamma_\pm^2\right)\left(\frac{1-v_\pm^2}{\rho}T'(v_\pm,\rho) + 2K_1(\rho)\right)\right. \\
&\qquad \left. - 2\gamma_\pm^2 v_\pm^2 K_1(\rho) - \frac{2}{\rho}\gamma_\pm^2 v_\pm[T(v_\pm,\rho) + 2v_\pm K_0(\rho)]\right\},
\end{aligned} \tag{42}$$

with $T'(v_\pm,\rho) = \partial T(v_\pm,\rho)/\partial v_\pm$, $\gamma_\pm = \left(1-v_\pm^2\right)^{-1/2}$, and with A given by (39). The sign $\sigma_\pm = \varepsilon_\pm/|\varepsilon_\pm|$ is needed in the LD region to take account of the fact that $\varepsilon_\pm$ can be negative while $\gamma_\pm = |\varepsilon_\pm|/m$ are positive by definition. The sign $\sigma_\pm$ is replaced by unity in the DL and PC regions. The RQPDFs (42) characterize the response of a nondegenerate thermal electron gas when RQ effects are included.

The interpretation of $v_\pm$ and $\gamma_\pm$ is different in the LD and PC regimes. In the LD region one has

$$v_\pm = \frac{\omega}{|\mathbf{k}|}\frac{\varepsilon_k \pm |\mathbf{k}|^2/2\omega}{\varepsilon_k \pm \omega/2}, \qquad \gamma_\pm = \left(\varepsilon_k \pm \frac{1}{2}\omega\right)\bigg/ m, \tag{43}$$

with ε_k defined by (12). In the nonquantum limit, one has $\varepsilon_k \gg \omega/2, |\mathbf{k}|^2/2\omega$, implying $v_\pm \to \omega/|\mathbf{k}| = z$. Then (42) reproduces the known nonquantum limit (Melrose and Hayes 1984):

$$S^{(0)}(k) = \frac{\omega A}{mz}[zT(z,\rho) + 2K_0(\rho)],$$

$$S^{(1)}(k) = \frac{2A}{\rho^2 z}[-zT(z,\rho) + (1-z^2)T'(z,\rho) + 2\rho K_1(\rho)], \tag{44}$$
$$S^{(2)}(k) = \frac{\omega A}{mz}\big[\gamma_0^2 zT(z,\rho) + 2\gamma_0^2(1+z^2)K_0(\rho) + K_2(\rho)\big].$$

In comparing (42) with (44), it is apparent that the phase speed, $z = \omega/|\mathbf{k}|$, in the nonquantum case is replaced by two functions $v_\pm$ that include the effect of the quantum recoil, which has opposite signs for emission and absorption. Thus, in the LD region, $v_\pm$ are interpreted as resonant phase speeds for stimulated emission and true absorption, which differ due to the quantum recoil, and $m\gamma_\pm$ are interpreted as the energies of the electron before and after emission of a wave quantum, respectively. In the PC regime, $m\gamma_\pm = \varepsilon_k \pm \frac{1}{2}\omega$ are interpreted as the energies of the created electron and positron.

In the DL regime the RQPDFs must be real. In this case, the $v_\pm$ are complex conjugates of each other. With $\sigma_\pm = 1$ and $T(v^*,\rho) = T^*(v,\rho)$, the sum over $\pm$ in (42) leads to a real expression, as required.

Comparison of the imaginary parts for the response functions obtained from the imaginary parts of the RQPDFs with those obtained by imposing the causal condition directly provides a check on both results. Here the imaginary parts that are to be compared are those obtained by inserting the imaginary parts of the $S^{(n)}(k)$ into expressions (10) and (11) for the longitudinal and transverse response, and those obtained by evaluating the integrals in (22), (23) and (24), (25) for the nondegenerate distribution (39). The results agree. In particular, Tystovich (1961) wrote explicit expressions for dissipation due to LD and PC in a nondegenerate electron gas, which our derivation reproduces: the spurious factor of 2 is present only in Tsytovich's calculation of the vacuum contribution.

6. Applications

RQ effects become important for dissipation and dispersion in plasmas only under extreme conditions, such as the early universe, quark–gluon plasmas and the interiors of compact stars. The degeneracy condition (temperature less than chemical potential) is relevant to only the last of these, and we concentrate on this case, emphasizing the role of PC. First we comment on LD.

LD in the nonquantum limit is possible only for subluminal waves, $\omega < |\mathbf{k}|$, and this is also the case when RQ effects are included. This precludes LD of transverse waves, which are superluminal. Dissipation and dispersion associated with LD may be treated nonrelativistically provided not only that the particles are nonrelativistic, but also that the waves are subluminal. The RQ recoil term changes the classical resonance condition to $\omega - \mathbf{k}\cdot\mathbf{v} \pm (\omega^2 - |\mathbf{k}|^2)/2m\gamma$, for emission and absorption, whereas in nonrelativistic theory, the recoil term is $\mp|\mathbf{k}|^2/2m$. The difference is unimportant for nonrelativistic particles and subluminal waves, but is important for superluminal waves and for waves with near vacuum dispersion, $\omega \approx |\mathbf{k}|$. An implication is that the widely used response functions of Lindhard (1954), which were derived using nonrelativistic quantum mechanics, may lead to unreliable results for waves with $\omega \gtrsim |\mathbf{k}|$. We are currently investigating this point.

The contribution of the plasma process to neutrino emission from the cores of compact stars depends on the dispersive properties of the degenerate gas, and Braaten (1991) pointed out that earlier authors, following Baudet *et al* (1971), had used inaccurate forms for the dispersion relations. A controversial point was raised by Braaten (1991), who criticized the claim by Baudet *et al* (1971), and subsequent authors, that PC needs to be taken into account in sufficiently hot and dense plasmas: if PC is allowed then photons decay into pairs much faster than they would decay into neutrinos. This point was discussed further by Itoh *et al* (1992) and Braaten and Segel (1993), who also concluded that PC is forbidden in a completely degenerate electron gas. In more recent discussions (Ratkoviȁ *et al* 2003, Dutta *et al* 2004, Koers and Wijers 2005, Jaikumar *et al* 2005), the approximations of Braaten and Segel (1993) to the dispersion functions have been used. Our results show that PC is allowed in a completely degenerate electron gas, and the reason that our results differ from those of these earlier authors can be understood as follows.

Itoh *et al* (1992) considered wave quanta at the cutoff frequency, ω_c, and argued that although one can have $\omega_c > 2m$ in a superdense plasma, the actual threshold for PC is $2\varepsilon_{\rm F}$, and one cannot have $\omega_c > 2\varepsilon_{\rm F}$. The higher threshold is because all electron states below the Fermi energy are occupied, and for $|\mathbf{k}| = 0$ the electron and positron energies are equal. This argument is consistent with our results, but it applies only at $|\mathbf{k}| = 0$. PC is forbidden in region (e) in figure 1, and this region shrinks as $|\mathbf{k}|$ increases. A dispersion curve for transverse waves that starts at $\omega_c > 2m$ for $|\mathbf{k}| = 0$ is necessarily in region (e) for sufficiently small $|\mathbf{k}|$, but then necessarily enters region (f), where PC is allowed, before approaching the light line asymptotically. Braaten and Segel (1993) made approximations to the wave dispersion by neglecting the quantum recoil: specifically, if one combines the denominators in (1), the common factor may be written as $(ku)^2 - (k^2/2m)^2$, and Braaten and Segel (1993) argued that for practical purposes one can neglect the $(k^2/2m)^2$ term. However, near the cutoff $|\mathbf{k}| \to 0$, one has $(ku)^2 - (k^2/2m)^2 \to \omega_c^2\gamma^2\big(1 - \omega_c^2\big/4m^2\gamma^2\big)$, and their approximation requires $\omega_c \ll 2m\gamma$, which is not satisfied for all $1 < \gamma < \varepsilon_{\rm F}/m$ in a superdense plasma. This approximation effectively excludes dispersion due to PC, and it is inconsistent to use it to argue that PC cannot occur. Our results show that PC does occur over a limited range of $|\mathbf{k}|$ in a completely degenerate electron gas, but not in region (e) in figure 1, due to exact cancellation of the vacuum contribution to PC by the electron gas. For a partially degenerate electron gas, even in region (e) the cancellation is not exact, and PC occurs.

The original argument of Braaten (1991) against PC was based on mass renormalization of the electron suppressing the cutoff frequency and keeping it below the PC threshold. We do not comment specifically on this argument here. Our conclusion is that the arguments by Itoh *et al* (1992) and Braaten and Segel (1993) that neglected mass renormalization do not negate the original argument of Baudet *et al* (1971) that PC needs to be taken into account when considering the plasma process for neutrino emission in a superdense plasma.

7. Conclusions

In this paper we discuss the properties of RQPDFs for an isotropic, unmagnetized plasma. The dispersion is related to the dissipation, which includes the familiar Landau damping (LD), modified by the quantum recoil, and one-photon pair creation (PC). It is necessary to treat the dissipation in the LD and PC regimes differently, and to interpret them differently. LD has the same interpretation as in a nonquantum plasma, except that the resonance at the phase speed, $z = \omega/|\mathbf{k}|$, is replaced by resonances at two speeds, $v_\pm$, and corresponding energies, $\varepsilon_\pm$, given by (43) and interpreted as the resonant values for induced emission and true absorption when the quantum recoil is included. Dissipation due to PC in the electron gas has the opposite sign to LD and a different interpretation: PC exists in the vacuum, due to the imaginary part of the vacuum polarization tensor (26), and the presence of an electron gas partly suppresses PC due to the Pauli exclusion principle.

An objective in this paper is to relate dissipation and dispersion by deriving the dissipation from the imaginary parts of the RQPDFs, which requires that the imaginary parts be determined explicitly. In particular, the logarithmic RQPDFs that appear for a completely degenerate electron gas acquire an imaginary part of $\pm i\pi$ when their arguments become negative, and a prescription is needed to determine the sign of this imaginary part uniquely. We start from the DL region, where the imaginary part is necessarily zero, and analytically continue into the regions where LD and PC are allowed. We show that the Landau prescription leads to a relatively simple prescription for identifying the sign of the imaginary part acquired when the argument of the logarithm changes sign. We compare our results with existing expressions for the imaginary parts derived in other ways, and find agreement provided that some minor errors are corrected. Specific errors identified are a spurious multiplicative factor in Jancovici's (1962) transverse response function and a factor of 2 in the expression derived by Tsytovich (1961) for the vacuum contribution to dissipation due to PC.

In the absence of any plasma, dissipation due to PC is determined by the imaginary part of the vacuum polarization tensor for $\omega > (m^2 + |\mathbf{k}|^2)^{1/2}$, and is zero otherwise. The presence of an electron gas tends to suppress PC, and the presence of a completely degenerate electron gas can completely suppress PC. Complete suppression at a given ω, $|\mathbf{k}|$ occurs if all potential states for the created electron are below the Fermi level. Although it was argued by Itoh *et al* (1992) and Braaten and Segel (1993) that PC cannot occur in a superdense plasma, where the cutoff frequency exceeds the PC threshold $2m$, we show this is not the case for at least a range of $|\mathbf{k}| \neq 0$. The earlier arguments of Baudet *et al* (1971) on the implications of PC remain valid and need to be taken into account in detailed analyses (e.g., Jaikumar *et al* (2005)).

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References

Baudet G, Petrosian V and Salpeter E E 1971 *Astrophys. J.* **150** 979
Berestetskii V B, Lifshitz E M and Pitaevskii L P 1971 *Relativistic Quantum Theory* (Oxford: Pergamon)
Braaten E 1991 *Phys. Rev. Lett.* **66** 1655
Braaten E 1992 *Astrophys. J.* **392** 70
Braaten E and Segel D 1993 *Phys. Rev.* D **48** 1478
Delsante A E and Frankel N E 1980 *Ann. Phys. (NY)* **125** 135
Dutta S I, Ratkoviċ S and Prakash M 2004 *Phys. Rev.* D **69** 023005
Godfrey B B, Newberger B S and Taggart K A 1975 *IEEE Trans. Plasma Sci.* **3** 60
Hakim R 1978 *Riv. Nuovo Cimento* **1** 1
Hakim R and Heyvaerts J 1978 *Phys. Rev.* A **18** 1250
Hakim R and Heyvaerts J 1980 *J. Phys. A: Math. Gen.* **13** 2001
Hayes L M and Melrose D B 1984 *Aust. J. Phys.* **37** 615
Itoh N, Mutoh H, Hikita A and Kohyama Y 1992 *Astrophys. J.* **395** 622
Jaikumar P, Gale C and Page D 2005 *Phys. Rev.* D **72** 123004
Jancovici B 1962 *Nuovo Cimento* **25** 428
Koers H B J and Wijers R A M J 2005 *Mon. Not. R. Astron. Soc.* **364** 934
Kowalenko V, Frankel N E and Hines K C 1985 *Phys. Rep.* **126** 109
Lindhard D J 1954 *Mat.-Fys. Medd. K. Dan. Vidensk. Selsk.* **28** 1
Melrose D B and Hayes L M 1984 *Aust. J. Phys.* **37** 639
Ratkoviċ S, Dutta S I and Prakash M 2003 *Phys. Rev.* D **67** 123002
Sivak H D 1985 *Ann. Phys.* **159** 351
Tsytovich V N 1961 *Sov. Phys. JETP* **13** 1249
Williams D R M and Melrose D B 1989 *Aust. J. Phys.* **42** 59